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Oscillating Tchebycheff Systems*

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As is well known the Tchebycheff polynomial of degree n minimizes the sup norm over all monic polynomials with n simple zeros in $[-1, +1]$. B. D. Bojanov [*J. Approx. Theory* **26** (1979), 293–300] recently investigated the situation for polynomials with a full set of zeros of higher multiplicities. In this paper we generalize these results to extended complete Tchebycheff systems.

INTRODUCTION

Recently in an interesting paper Bojanov [2] extended the concept of a Tchebycheff polynomial to the case where the polynomials to be analyzed may have zeros of order larger than one. We propose to generalize this result to extended complete Tchebycheff systems. Here one does not have the factoring properties of polynomials which play such a prominent role in the proof for polynomials.

Specifically let $\{\mu_i + 1\}_{i=1}^{K}$ be a set of positive integers where $N+1 = \sum_{i=1}^{K}(\mu_i + 1)$ and let $\{v_i\}_{i=0}^{N+1}$ form an extended complete Tchebycheff system of order $M+1 = \max_{1 \leqslant j \leqslant K}\{\mu_j\}$ with $v_0(x) \equiv 1$. This means that, for each i with $0 \leqslant i \leqslant N+1$, $\{v_j\}_{j=0}^{i} \subset C^M[0,1]$ form an extended Tchebycheff system of order $M+1$; that is, $v = \sum_{j=0}^{i} a_j v_j$ and $\sum_{j=0}^{i} a_j^2 > 0 \Rightarrow v$ has at most i zeros in $[0, 1]$ counting multiplicities up to order $M+1$ and, for $0 \leqslant x_0 < x_1 < \cdots < x_i \leqslant 1$,

$$V\begin{pmatrix} 0, 1,\ldots, i \\ x_0,\ldots, x_i \end{pmatrix} = \det\{v_s(x_j); 0 \leqslant s, j \leqslant i\} > 0.$$

Define Δ_K to be the open simplex,

$$\Delta_K = \{\mathbf{z} = (z_1,\ldots, z_K)\colon 0 < z_1 < \cdots < z_K < 1\}.$$

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For each $\mathbf{z} \in \Delta_K$, let $v(\cdot\,; \mathbf{z})$ be the unique element of the form

$$v = v_{N+1} + \sum_{i=0}^{N} a_i v_i \tag{1a}$$

for which

$$\frac{d^j v}{dx^j}(z_i) = 0 \qquad (j = 0, 1,..., \mu_i)\ (i = 1,..., K). \tag{1b}$$

If $\|f\| = \max_{x \in [0,1]} |f(x)|$, the main problem we consider is how to characterize the solution of

$$\min_{\mathbf{z} \in \Delta_K} \|v(\cdot, \mathbf{z})\|. \tag{1c}$$

The raison d'être for wanting to solve (1) is provided by the following interpolation problem. For each $\mathbf{z} \in \Delta_K$ and $f \in C^M[0, 1]$ let v be the unique element in the subspace spanned by $\{v_i\}_{i=0}^N$ which satisfies

$$v^{(j)}(x_i) = f^{(j)}(x_i) \qquad (j = 0, 1,..., \mu_1)\ (i = 1,..., K).$$

Then it is well known that

$$f(x) - v(x) = f[\overbrace{z_1,..., z_1}^{\mu_1+1}, z_2, z_2,..., z_{K-1}, \overbrace{z_K,..., z_K}^{\mu_K+1}]\, v(x, \mathbf{z}),$$

for $x \in [0, 1]$, where $f[z_1,..., z_K]$ is a generalized divided difference of f with respect to the system $\{v_i\}_{i=0}^{N+1}$. Hence it is clear that a good choice for the interpolating points is the set $\mathbf{z}$ which minimizes (1c). For the case of polynomials with all $\mu_i = 0$, one is referred to [3, 6] for a detailed analysis.

Oscillating Systems

We consider first a more general problem. Namely, we are given a set of positive integers $\{m_i\}_{i=1}^n$; an extended complete Tchebycheff system $\{v_i\}_{i=0}^{N+1} \subset C^S[0, 1]$ of order $S + 1 = \max_{1 \leq i \leq n} m_i$ with $v_0 \equiv 1$ and with $N = \sum_{i=1}^n m_i$; and a set of real numbers $\{d_i\}_{i=0}^{n+1}$ which yield a sequence

$$e_i = d_i - d_{i-1} \qquad [i = 1,..., n + 1)$$

with the property $\operatorname{sgn} e_i = (-1)^{N + \sum_{j=1}^{i-1} m_j} = (-1)^{\sum_{j=i}^{n} m_j}$ and, in particular, $e_{n+1} > 0$. Let

$$\Delta_n = \{\mathbf{x} = (x_1,..., x_n): 0 < x_1 < \cdots < x_n < 1\}.$$

Then among all functions of the form

$$v = v_{N+1} + \sum_{i=1}^{N} a_i v_i + a_0, \tag{2}$$

we seek to find one with the property that for some $(x_1,..., x_n) \in \Delta_n$ and an $E \in R^+$,

$$v(x_i) = Ed_i \qquad (i = 0, 1,..., n+1), \tag{3a}$$

$$v^{(j)}(x_i) = 0 \qquad (j = 1,..., m_i; i = 1,..., n), \tag{3b}$$

where $x_0 \equiv 0$ and $x_{n+1} \equiv 1$.

The purpose of this section is to show that there is exactly one solution to this problem.

Set

$$u_i(x) = \frac{d}{dx} v_i(x) \qquad (i = 1,..., N+1).$$

From [5, p. 379], it follows that $\{u_i\}_{i=1}^{N+1}$ form an extended complete Tchebycheff system on [0, 1]. Thus since the system

$$v(x_0) = Ed_0,$$

$$v(x_k) - v(x_{k-1}) = Ee_k \qquad (k = 1,..., n+1),$$

$$v^{(j)}(x_k) = 0 \qquad (j = 1,..., m_k)\ (k = 1,..., n)$$

is equivalent to (3), we can rephrase the problem as follows: Find a function of the form,

$$u = u_{N+1} + \sum_{i=1}^{N} a_i u_i \tag{4}$$

such that for some $x = (x_1,..., x_n) \in \Delta_n$, and an $E > 0$,

$$\int_{x_{k-1}}^{x_k} u(x)\, dx = Ee_k \qquad (k = 1,..., n+1), \tag{5a}$$

$$u^{(j)}(x_k) = 0 \quad (j = 0, 1,..., m_k - 1; k = 1,..., n.) \tag{5b}$$

Hence u satisfies (5a), (5b) iff the function v given by

$$v(x) = Ed_0 + \int_{x_0}^{x} u(x)\, dx \tag{5c}$$

satisfies (3).

For the extended definition

$$U^*\begin{pmatrix}u_1,\dots,u_{N+1}\\x_1,\dots,x_{N+1}\end{pmatrix}$$

of the determinant

$$U\begin{pmatrix}u_1,\dots,u_{N+1}\\x_1,\dots,x_{N+1}\end{pmatrix}=\det\{u_i(x_j);\ i,j=1,\dots,N+1\};$$

in case of coincidences among the x_i, one is referred to [5, p. 5].

LEMMA 1. *For each* $\mathbf{x}=(x_1,\dots,x_n)\in\Delta_n$ *there is a unique u of the form* (4) *which we designate by* $u(\cdot,\mathbf{x})$ *which satisfies* (5b).

Proof. Set

$$u(x,\mathbf{x})=\frac{U^*\begin{pmatrix}u_1, & \cdots, & \cdots, & \cdots, & u_N, & u_{N+1}\\ \bar{x}_1, & \cdot & \cdot & \cdot & \bar{x}_N, & x\end{pmatrix}}{U^*\begin{pmatrix}u_1, & \cdots, & u_N\\ \bar{x}_1, & \cdots, & \bar{x}_N\end{pmatrix}} \tag{6}$$

with $\bar{x}_1,\dots,\bar{x}_N$ the sequence obtained from $x_1,\dots,x_n$ by repeating x_i, m_i times, $i=1,\dots,n$. It is easy to verify that $u(\cdot,\mathbf{x})$ has the desired properties. For uniqueness one notes that the difference of two solutions must lie in the liner span of $\{u_i\}_{i=1}^N$. Thus if the difference is non-zero it can have at most $N-1$ zeros including multiplicities. Hence uniqueness follows. ∎

Remark 1. For $\mathbf{x}=(x_1,\dots,x_n)\in\Delta_n$, and $x\notin\mathbf{x}$ differentiating (6) one finds

$$\begin{aligned}\frac{\partial}{\partial x_i}u(x,\mathbf{x})&=\frac{1}{U^*\begin{pmatrix}u_1,\dots,u_N\\ \bar{x}_1,\dots,\bar{x}_N\end{pmatrix}^2}\\&\quad\times\left[U_i^*\begin{pmatrix}u_1,\dots,u_{N+1}\\ \bar{x}_1,\dots,\bar{x}_n,x\end{pmatrix}\cdot U^*\begin{pmatrix}u_1,\dots,u_N\\ \bar{x}_1,\dots,\bar{x}_N\end{pmatrix}\right.\\&\quad\left.-U_i^*\begin{pmatrix}u_1,\dots,u_N\\ \bar{x}_1,\dots,\bar{x}_N\end{pmatrix}\cdot U^*\begin{pmatrix}u_1,\dots,u_{N+1}\\ \bar{x}_1,\dots,\bar{x}_N,x\end{pmatrix}\right]\end{aligned} \tag{7}$$

with the subscript i indicating that all the terms $u_r^{(m_i-1)}(x_i)$ appearing in $U^*(\ )$ have been replaced by $u_r^{(m_i)}(x_i)$.

We record several by-products of (7):

$$\frac{d^j}{dx^j}\left[\frac{\partial}{\partial x_i}u(x,\mathbf{x})\right]\Bigg|_{x=x_k}=0$$
$$(j=0,1,\ldots,m_k-2)\ (k=1,\ldots,n), \tag{8}$$

$$\frac{d^{m_k-1}}{dx^{m_k-1}}\left[\frac{\partial}{\partial x_i}u(x,\mathbf{x})\right]\Bigg|_{x=x_k}=0$$
$$(k=1,2,\ldots,i-1,i+1,\ldots,n), \tag{9}$$

$$\left[(-1)^{1+\sum_{j=i+1}^{n}m_j}\right]\frac{d^{m_i-1}}{dx^{m_i-1}}\left[\frac{\partial}{\partial x_i}u(x,\mathbf{x})\right]\Bigg|_{x=x_i}>0$$
$$(i=1,\ldots,n). \tag{10}$$

$(\partial/\partial x_i)\,u(x,\mathbf{x})$ is of the form

$$\frac{\partial}{\partial x_i}u(x,\mathbf{x})=\sum_{j=1}^{N}a_{ij}u_j(x)\qquad (i=1,\ldots,n). \tag{11}$$

In our later analysis, the Jacobian of the system (5) will be studied. In this investigation the signs of certain sub-determinants will play a key role.

LEMMA 2. *For* $1\leqslant k\leqslant n+1$ *and* $\mathbf{x}\in\Delta_n$ *set*

$$D_k(\mathbf{x})=\det D\begin{pmatrix}0,\ldots,k-1,k+1,\ldots,n\\1,\ldots\ldots\ldots\ldots\ldots\ldots,n\end{pmatrix} \tag{12}$$

with

$$D\begin{pmatrix}0,\ldots,n\\1,\ldots,n\end{pmatrix}=\left(\int_{x_i}^{x_{i+1}}(\partial u/\partial x_j)(x,\mathbf{x})\,dx\colon i=0,\ldots,n;j=1,\ldots,n\right).$$

Then

$$\operatorname{sgn}(D_k(\mathbf{x})\,e_k)=(-1)^{n-k-1}(-1)^{\sum_{i=1}^{n}\sum_{j=i}^{n}m_j}\qquad (k=1,\ldots,n+1).$$

Proof. We first claim that $D_k(\mathbf{x})\neq 0$. For if it was zero then there would be a function u of the form

$$u=\sum_{i=1}^{n}a_i\frac{\partial}{\partial x_i}u(\cdot,\mathbf{x}),\qquad \text{where}\quad \sum_{i=1}^{n}a_i^2>0$$

such that

$$\int_{x_{j-1}}^{x_j}u(x)\,dx=0\qquad (j=1,\ldots,k-1,k+1,\ldots,n+1). \tag{13}$$

From (8), (9), (10) we conclude that

$$u^{(\ell)}(x_j) = 0 \qquad (\ell = 0, 1, \ldots, m_j - 2)\ (j = 1, \ldots, n) \tag{14}$$

and that u is not identically zero. Now from (13) it follows that u must also have a zero in (x_{j-1}, x_j) $(j = 1, \ldots, k-1, k+1, \ldots, n+1)$. Thus u has at least N zeros (including multiplicities), which is a contradiction since u is a non-zero function which is in the span of $\{u_i\}_{i=1}^{N}$. We conclude that $D_k(\mathbf{x}) \neq 0$.

Next we note that this same argument implies also that if

$$[y_{j-1}, y_j] \subset [x_{j-1}, x_j] \qquad (j = 1, \ldots, k-1, k+1, \ldots, n+1)$$

and we replace x_j by y_j $(j = 0, 1, \ldots, n+1)$ in (12), the resulting determinant is non-zero. By continuity, it is clear that the sign of the determinant remains constant under these modifications.

Let the intervals of integration in (12) vary in the manner

$$\begin{gathered}[y_0, x_1] \to \{x_1\},\ [y_1, x_2] \to \{x_2\}, \ldots, [y_{k-2}, x_{k-1}] \to \{x_{k-1}\},\\ [x_k, y_{k+1}] \to \{x_k\},\ [x_{k+1}, y_{k+2}] \to \{x_{k+1}\}, \ldots, [x_n, y_{n+1}] \to \{x_n\}.\end{gathered} \tag{15}$$

From (8), (9), (10) we can infer that as the intervals get very small, the modified determinant, $\hat{D}_k$, has the property

$$\operatorname{sgn} \hat{D}_k = \operatorname{sgn}\left[\prod_{j=1}^{k-1} \int_{y_{j-1}}^{x_j} \frac{\partial u}{\partial x_j}(x, \mathbf{x})\, dx \prod_{j=k}^{n} \int_{x_j}^{y_{j+1}} \frac{\partial u}{\partial x_j}(x, \mathbf{x})\, dx\right]. \tag{16}$$

From (10)

$$\frac{\partial u}{\partial x_i}(x, \mathbf{x}) \sim c_i(-1)^{1 + \sum_{j=i+1}^{n} m_j}(x - x_i)^{m_i - 1}, \tag{17}$$

where $c_i > 0$ for x near x_i $(i = 1, \ldots, n)$. Hence

$$\begin{aligned}\operatorname{sgn} \hat{D}_k &= \left(\prod_{i=1}^{k-1} (-1)^{1 + \sum_{j=i+1}^{n} m_j}(-1)^{m_i - 1}\right)\left(\prod_{i=k}^{n} (-1)^{1 + \sum_{j=i+1}^{n} m_j}\right)\\ &= (-1)^{n-k+1}(-1)^{\sum_{i=1}^{n} \sum_{j=i+1}^{n} m_j}(-1)^{\sum_{i=1}^{k-1} m_i}\end{aligned}$$

and

$$\operatorname{sgn}[e_k D_k(\mathbf{x})] = (-1)^{n-k-1}(-1)^{\sum_{i=1}^{n} \sum_{j=i}^{n} m_j}. \quad \blacksquare$$

For a given $\mathbf{x} \in \Delta_n$, consider the system of $n+1$ differential equations

$$\frac{d}{ds}\left[\int_{x_{k-1}(s)}^{x_k(s)} u(x, \mathbf{x}(s))\, dx\right] = e_k \frac{dE(s)}{ds} - f_k \qquad (k = 1, \ldots, n+1) \tag{18}$$

in the $n+1$ variables $(\mathbf{x}(s), E(s))$ with the initial conditions: $\mathbf{x}(0)=\mathbf{x}=(x_1,\ldots,x_n)$ and $E(0)=0$, where

$$\int_{x_{k-1}}^{x_k} u(x,\mathbf{x})\,dx = f_k \qquad (k=1,\ldots,n+1).$$

Note that if we integrate (18) we find that

$$\int_{x_{k-1}(s)}^{x_k(s)} u(x,\mathbf{x}(s))\,dx = e_k E(s) + (1-s) f_k \tag{19}$$

and thus at $s=1$ we get the desired solution to (5). Our problem is to show that the solution to (18) can be extended to $[0,1]$. The arguments needed to demonstrate this follow the pattern established in [4]. Several things have to be verified. We proceed to do this.

First, (18) can be written as

$$\sum_{j=1}^{n}\left[\int_{x_{k-1}(s)}^{x_k(s)} \frac{\partial u(x,\mathbf{x}(s))}{\partial x_j(s)}\right]\frac{dx_j(s)}{ds} - e_k\frac{dE(s)}{ds} = -f_k \qquad (k=1,\ldots,n+1). \tag{19a}$$

The determinant of the left hand side of this system is

$$\sum_{k=1}^{n+1} D_k(\mathbf{x})(-1)^{n+k} e_k \tag{20}$$

which by Lemma 2 is non-zero. Next it is easy to check [see (6)] that

$$e_k f_k > 0 \qquad (k=1,\ldots,n+1).$$

Thus solving (19a) for dE/ds, one finds

$$0 < \frac{\min_k |f_k|}{\max_k |e_k|} \leqslant \frac{dE}{ds} = \frac{\sum (-1)^{n+k} f_k D_k(\mathbf{x})}{\sum (-1)^{n+k} e_k D_k(\mathbf{x})} \leqslant \frac{\max_k |f_k|}{\min_k |e_k|}. \tag{21}$$

From (21) it follows that $E(s)$ is bounded, positive, and monotone increasing as s varies over $(0,1]$. Hence from (19) and the fact that $e_k f_k > 0$ we see that there is no sequence of solutions $\{\mathbf{x}(s_\nu), E(s_\nu)\}$ where $\{s_\nu\} \subset (0,1]$ so that $\mathbf{x}(s_\nu) \to \partial\Delta_n$.[1] Thus the essential properties have been verified and the solution can be extended to $[0,1]$ (for more details see [1, 4]).

Before proceeding with the uniqueness portion of the proof it is interesting

[1] By the usual argument since $u(x,\mathbf{x}(s_\nu))$ has a full set of zeros, its coefficients are bounded.

to note that integrating (21) between 0 and 1 yields a generalized de la Vallée Poussin system of inequalities.

$$\frac{\min_k |f_k|}{\max_k |e_k|} \leqslant E(1) \leqslant \frac{\max_k |f_k|}{\min_k |e_k|}.$$

For each $\mathbf{x} \in \Delta_n$, let $F(\mathbf{x}) = (\mathbf{x}(1), E(1))$, that is, the solution to the differential equations at $s = 1$ with the initial condition $\mathbf{x}(0) = \mathbf{x}$ and $E(0) = 0$. Since Δ_n is connected and since by the theory of differential equations, F is continuous, $F(\Delta_n)$ is connected. Further if $(\mathbf{x}, E)$ is a solution to (5) then it is easy to check that $\mathbf{x}(s) \equiv \mathbf{x}$ and $E(s) = sE$ is a solution to the system of differential equations with initial condition $\mathbf{x}(0) = \mathbf{x}$ and $E(0) = 0$ and by uniqueness it is the only solution. Hence F maps Δ_n onto W where

$$W = \{(\mathbf{x}, E): (\mathbf{x}, E) \text{ solves } (5),\ E > 0\}.$$

Thus W is connected. For each $(\mathbf{x}, E) \in W$ the Jacobian of the system (5) evaluated at $(\mathbf{x}, E)$ is non-zero by (20). The *Implicit Function Theorem* implies then that each point of W is an isolated point. Hence W consists of just one point; that is, the solution to (5) is unique.

Summarizing,

THEOREM 1. *For a given set of positive integers* $\{m_i\}_{i=1}^n$ $N = \sum_{i=1}^n m_i$ *and a set of* $\{d_i\}_{i=0}^{n+1}$ *where the corresponding* $\{e_i\}_{i=1}^{n+1}$ *satisfy* $e_i(-1)^{\sum_{j=i}^n m_j} > 0$ *there is a unique* $v(x)$ *of the form* (2) *such that for some unique* $\mathbf{x} = (x_1, ..., x_n) \in \Delta_n$, v *satisfies* (3).

We now return to the main problem as defined in (1).

LEMMA 3. *There is a unique function* $v(\cdot, \mathbf{z}^*)$ *of the form* (1a) *which satisfies* (1b) *and for some set,*

$$0 = t_0 < t_1 < \cdots < t_{K-1} < t_K = 1,$$

also satisfies

$$\|v(\ , \mathbf{z}^*)\| = (-1)^{N+1-\sum_{j=1}^{i}(\mu_j+1)} v(t_i, \mathbf{z}^*) \qquad (i = 0, 1, ..., K) \quad (22)$$

with $0 = t_0 < z_1^* < t_1 \cdots < t_{K-1} < z_K^* < t_K = 1$, *with* $N = \sum_{j=1}^K (\mu_j + 1) - 1$.

Proof. In Theorem 1 for $i = 0, 1, ..., 2K$, set

$$\begin{aligned} d_i &= 0 && \text{if } i \text{ is odd} \\ &= (-1)^{N+1-\sum_{j=1}^{i/2}(\mu_j+1)} && \text{if } i \text{ is even} \end{aligned}$$

and $n = 2K - 1$.

Further define for $i = 1,\ldots, 2K-1 = n$

$$m_i = \mu_{(i+1)/2} \quad \text{if} \quad i \text{ is odd}$$
$$= 1 \quad \text{if} \quad i \text{ is even.}$$

Finally delete all i such that $m_i = 0$. If $m_{i_1},\ldots, m_{i_\ell}$ are the remaining multiplicities where $i_1 < i_2 < \cdots < i_\ell$, then let $m_j = m_{i_j}$, $j = 1 \cdots \ell$. $N = \sum_{j=1}^{\ell} m_j$.

It is easy to verify that the resulting $\{e_i\}$ have the property that $e_i(-1)^{\sum_{j=i}^{\ell} m_j} > 0$. The result follows directly from Theorem 1, since $(d/dx)\, v(x, z^*)$ has at most N zeros.

THEOREM 2. *Among all elements of the form* (1a) *which satisfy* (1b) *there is exactly one element of minimal norm. This element is the unique function* $v(\cdot, \mathbf{z}^*)$ *which satisfies* (22).

Proof. Let $v(\cdot, \mathbf{z})$ be a candidate for the function of minimal norm. By Rolle's theorem there is a t_i where $z_i < t_i < z_{i+1}$ so that

$$\frac{dv}{dx}(x, \mathbf{z})\bigg|_{x=t_i} = 0 \qquad (i = 1,\ldots, K-1).$$

Set $\mathbf{x} = (x_1,\ldots, x_{2K-1}) = (z_1, t_1, z_2,\ldots, t_{K-1}, z_K)$. One now sees that $(d/dx)\, v(x, \mathbf{z})$ satisfies (5b) for the corresponding $\mathbf{x}$ with the corresponding m_i as in Lemma 3.

Using this $\mathbf{x}$, and

$$f_i = v(x_i, \mathbf{z}) - v(x_{i-1}, \mathbf{z}) \qquad (i = 1,\ldots, 2K)$$

as initial conditions for the differential equation (18), with the e_i and m_i as in Lemma 3, Eq. (21) becomes

$$\frac{dE}{ds} = \frac{\sum |f_k|\, |D_k(\mathbf{x}(s))|}{\sum |e_k|\, |D_k(\mathbf{x}(s))|} = \frac{E(1) \sum |f_k|\, |D_k(\mathbf{x}(s))|}{\sum E(1)\, |D_k(\mathbf{x}(s))|} \tag{23}$$

with $E(1) = \|v(\ ,\mathbf{z}^*)\|$. Note that Theorem 1 implies that the solution of (18) only depends on the e_i and m_i, not on the initial conditions; hence if $|f_k| < E(1)$ for some k, we would obtain the contradiction that $\int_0^1 (dE/ds)\, ds < E(1)$. Thus $f_k = e_k$, $k = 1,\ldots, 2K$, and uniqueness follows from Theorem 1.

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